Partitions of $\mathbb{Z}_n$ into Arithmetic Progressions

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Abstract

We introduce the notion of arithmetic progression blocks or AP-blocks of $\mathbb{Z}_n$, which can be represented as sequences of the form $(x, x+m, x+2m, \ldots, x+(i-1)m) \pmod n$. Then we consider the problem of partitioning $\mathbb{Z}_n$ into AP-blocks for a given difference m. We show that subject to a technical condition, the number of partitions of $\mathbb{Z}_n$ into m-AP-blocks of a given type is independent of m. When we restrict our attention to blocks of sizes one or two, we are led to a combinatorial interpretation of a formula recently derived by Mansour and Sun as a generalization of the Kaplansky numbers. These numbers have also occurred as the coefficients in Waring's formula for symmetric functions.

Keywords: Kaplansky number, cycle dissection, m-AP-partition, separation algorithm.

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1 Introduction

Let $\mathbb{Z}_n$ be the cyclic group of order n whose elements are written as $1, 2, \ldots, n$. Intuitively, we assume that the elements $1, 2, \ldots, n$ are placed clockwise on a cycle. Thus $\mathbb{Z}_n$ can be viewed as an *n-cycle*, more specifically, a directed cycle. In his study of the ménages problem, Kaplansky [7] has shown that the number of ways of choosing k elements from $\mathbb{Z}_n$ such that no two elements differ by one modulo n (see also Brauldi [1], Comtet [3], Riordan [14], Ryser [15] and Stanley [16, Lemma 2.3.4]) equals

$$\frac{n}{n-k}\binom{n-k}{k}. \tag{1.1}$$

Moreover, Kaplansky [8] considered the following generalization. Assume that $n \geq pk+1$. Then the number of k-subsets $\{x_1, x_2, \ldots, x_k\}$ of $\mathbb{Z}_n$ such that

$$x_i - x_j \notin \{1, 2, \ldots, p\} \tag{1.2}$$

for any pair (x_i, x_j) of distinct elements, is given by

$$\frac{n}{n-pk}\binom{n-pk}{k}. \tag{1.3}$$

Here we clarify the meaning of the notation (1.2). Given two elements x and y of $\mathbb{Z}_n$, $x-y$ may be considered as the distance from y to x on the directed cycle $\mathbb{Z}_n$. Therefore, (1.2) says that the distance from any element x_i to any other element x_j on the directed cycle $\mathbb{Z}_n$ is at least $p+1$.

From a different perspective, Konvalina [10] studied the number of k-subsets $\{x_1, x_2, \ldots, x_k\}$ such that no two elements x_i and x_j are "uni-separated", namely $x_i - x_j \neq 2$ for all x_i and x_j. Remarkably, Konvalina discovered that the answer is also given by the Kaplansky number (1.1) for $n \geq 2k+1$. Other generalizations and related questions have been investigated by Hwang [5], Hwang, Korner and Wei [6], Munarini and Salvi [12], Prodinger [13] and Kirschenhofer and Prodinger [9]. Recently, Mansour and Sun [11] obtained the following unification of the formulas of Kaplansky and Konvalina.

Theorem 1.1. *Assume that $m, p, k \geq 1$ and $n \geq mpk+1$. The number of k-subsets $\{x_1, x_2, \ldots, x_k\}$ of $\mathbb{Z}_n$ such that*

$$x_i - x_j \notin \{m, 2m, \ldots, pm\} \tag{1.4}$$

for any pair (x_i, x_j), is given by the formula (1.3), and is independent of m.

In the spirit of the original approach of Kaplansky, Mansour and Sun first solved the enumeration problem of choosing k-subset from an n-set with elements lying on a line. They established a recurrence relation, and solved the equation by computing the residues of some Laurent series. The case for an n-cycle can be reduced to the case for a line. They raised the question of finding a combinatorial proof of their formula. Guo [4] found a proof by using number theoretic properties and Rothe's identity:

$$\sum_{k=0}^{n} \frac{xy}{(x+kz)(y+(n-k)z)}\binom{x+kz}{k}\binom{y+(n-k)z}{n-k} = \frac{x+y}{x+y+nz}\binom{x+y+nz}{n}.$$

This paper is motivated by the question of Mansour and Sun. We introduce the notion of arithmetic progression blocks or AP-blocks of $\mathbb{Z}_n$. A sequence of the form

$$(x, x+m, x+2m, \ldots, x+(i-1)m) \pmod{n}$$

is called an AP-block, or an m-AP-block, of length i and of difference m. Then we consider partitions of $\mathbb{Z}_n$ into m-AP-blocks $B_1, B_2, \ldots, B_k$ of the same difference m. The type of such a partition is referred to as the type of the multisets of the sizes of the blocks. Our main result shows that subject to a technical condition, the number

of partitions of $\mathbb{Z}_n$ into m-AP-blocks of a given type is independent of m and is equal to the multinomial coefficient.

This paper is organized as follows. In Section 2, we give a review of the cycle dissections and make a connection between the Kaplansky numbers and the cyclic multinomial coefficients. We present the main result in Section 3, that is, subject to a technical condition, the number of partitions of $\mathbb{Z}_n$ into m-AP-blocks of a given type equals the multinomial coefficient and does not depend on m. We present a separation algorithm which leads to a bijection between m-AP-partitions and m'-AP-partitions of $\mathbb{Z}_n$. The correspondence between m-AP-partitions and cycle dissections ($m' = 1$) implies the main result Theorem 3.2. For the type $1^{n-(p+1)k}(p+1)^k$ we are led to a combinatorial proof which answers the question of Mansour and Sun.

2 Cycle Dissections

In their combinatorial study of Waring's formula on symmetric functions, Chen, Lih and Yeh [2] introduced the notion of cycle dissections. Recall that a *dissection of an n-cycle* is a partition of the cycle into blocks, which can be viewed by putting cutting bars on some edges of the cycle. Note that there at least one bar to cut a cycle into straight segments. A dissection of an n-cycle is said of *type* $1^{k_1}2^{k_2}\cdots n^{k_n}$ if there are k_i blocks of i elements in it. For instance, Figure 1 gives a 20-cycle dissection of type $1^8 2^3 3^2$.

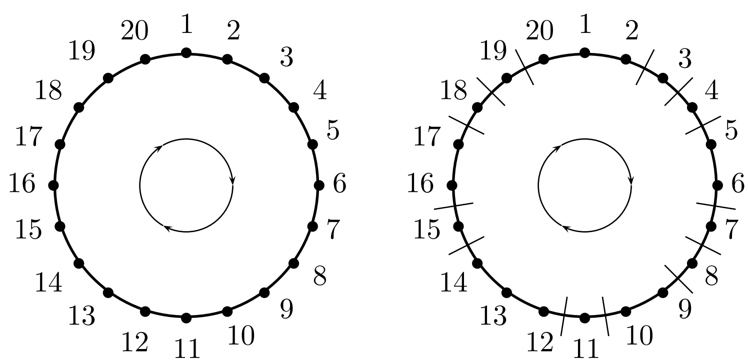


Figure 1: A 20-cycle dissection of type $1^8 2^3 3^2$.

The following lemma is due to Chen-Lih-Yeh [2, Lemma 3.1].

Lemma 2.1. *For an n-cycle, the number of dissections of type $1^{k_1}2^{k_2}\cdots n^{k_n}$ is given by the* cyclic multinomial coefficients*:*

$$\frac{n}{k_1 + \cdots + k_n}\binom{k_1 + \cdots + k_n}{k_1, \ldots, k_n}. \tag{2.1}$$

This lemma is easy to prove. Given a dissection, one may pick up any segment as a distinguished segment. This can be done in $k_1 + k_2 + \cdots + k_n$ ways. On the other hand, any of the n elements can serve as the first element of the distinguished segment.

Consider a cycle dissection of type $1^{n-(p+1)k}(p+1)^k$. The set of the first elements of each segment of length $p+1$ corresponds a k-subset of $\mathbb{Z}_n$ satisfying (1.2). Thus the cyclic multinomial coefficient of type $1^{n-(p+1)k}(p+1)^k$ reduces to (1.3) and particularly the cyclic multinomial coefficient of type $1^{n-2k}2^k$ reduces to the Kaplansky number (1.1).

3 Partitions of $\mathbb{Z}_n$ into Arithmetic Progressions

In this section, we present the main result of this paper, namely, a formula for the number of partitions of $\mathbb{Z}_n$ into m-AP-blocks of a given type. The proof is based on a separation algorithm to transform an m-AP-partition to an m'-AP-partition.

We begin with some concepts. First, $\mathbb{Z}_n$ is considered as a directed cycle. An arithmetic progression block, or an AP-block of $\mathbb{Z}_n$, is defined to be a sequence of elements of $\mathbb{Z}_n$ of the following form

$$B = (x, x+m, x+2m, \ldots, x+(i-1)m) \pmod{n},$$

where m is called the *difference* and i is called the *length* of B. An AP-block of difference m is called an m-AP-block. If B contains only one element, then it is called a *singleton*. The first element x is called the *head* of B. An m-AP-partition, or a partition of $\mathbb{Z}_n$ into m-AP-blocks, is a set of m-AP-blocks of $\mathbb{Z}_n$ whose underlying sets form a partition of $\mathbb{Z}_n$. For example,

$$(7, 9, 11),\ (8),\ (10, 12),\ (1),\ (2, 4, 6),\ (3),\ (5) \tag{3.1}$$

is a 2-AP-partition of $\mathbb{Z}_{12}$ with four singletons and three non-singleton heads 7, 10 and 2.

It should be noted that different AP-blocks may correspond to the same underlying set. For example, $(1, 3)$ and $(3, 1)$ are regarded as different AP-blocks of $\mathbb{Z}_4$, but they have the same underlying set $\{1, 3\}$. On the other hand, as will be seen in Proposition 3.1, it often happens that an AP-block is uniquely determined by its underlying set. For example, given the difference $m = 3$, the AP-block $(12, 15, 2, 5, 8)$ of $\mathbb{Z}_{16}$ is uniquely determined by the underlying set $\{2, 5, 8, 12, 15\}$ since there is only one way to order these five elements to form an arithmetic progression of difference 3 modulo 16.

For an m-AP-partition π, the *type* of π is defined by the type of the multisets of the sizes of the blocks. Usually, we use the notation $1^{k_1}2^{k_2}\cdots n^{k_n}$ to denote a type for which there are k_1 blocks of size one, k_2 blocks of size two, etc. However, for the sake of presentation, we find it more convenient to ignore the zero exponents and express a

type in the form $i_1^{k_1} i_2^{k_2} \cdots i_r^{k_r}$, where $1 \le i_1 < i_2 < \cdots < i_r$ and all $k_j \ge 1$. For example, the AP-partition (3.1) is of type $1^4 2^1 3^2$.

Throughout this paper, we restrict our attention to m-AP-partitions with at least one singleton block and also at least one non-singleton block, namely, $i_1 = 1$ and $r \ge 2$ in the above notation of types. Here is the aforementioned condition:

$$\left\lceil \frac{k_1}{k_2 + \cdots + k_r} \right\rceil \ge (m-1)(i_r - 1), \tag{3.2}$$

where the notation $\lceil x \rceil$ for a real number x stands for the smallest integer that larger than or equal to x. Obviously, the condition (3.2) holds for $m = 1$. For $m \ge 2$, (3.2) is equivalent to the relation

$$k_1 \ge (k_2 + \cdots + k_r)\big[(m-1)(i_r - 1) - 1\big] + 1. \tag{3.3}$$

We prefer the form (3.2) for a reason that will become clear in the combinatorial argument in the proof of Theorem 3.2. In fact on an n-cycle dissection, the $\sum_{j=2}^{r} k_j$ non-singleton heads divide the k_1 singletons into $\sum_{j=2}^{r} k_j$ segments. By virtue of the pigeonhole principle, there exists a segment containing at least $(m-1)(i_r-1)$ singletons.

For example in the AP-partition (3.1), the three non-singleton heads divide the four singletons into three segments and therefore there exists one segment containing at least 2 singletons. In this particular partition it is the path from 2 to 7 that contains two singletons 3 and 5, see the right cycle in Figure 2.

Proposition 3.1. *Under the condition* (3.2)*, an m-AP-block is not uniquely determined by its underlying set if and only if $n = i_r m$ and it is of length i_r.*

Proof. Let $n = i_r m$. Consider the AP-blocks,

$$B_j = (x + jm,\ x + (j+1)m,\ \ldots,\ x + (j + i_r - 1)m) \pmod{n}, \quad 0 \le j \le i_r - 1.$$

It is easy to see that these AP-blocks B_j $(j = 0, 1, \ldots, i_r - 1)$ have the same underlying set

$$\{x,\ x + m,\ \ldots,\ x + (i_r - 1)m\}.$$

Conversely, suppose that there is an m-AP-block B of length i_s which is not uniquely determined by its underlying set. We may assume that there exists another AP-block B' having the same underlying set as B. Thus the difference between B and B' lies only in the order of their elements as a sequence. It follows that $n = i_s m$ for some $2 \le s \le r$. If $m = 1$, then $n = i_s$ which yields $s = r = 1$, a contradiction. So we may assume that $m \ge 2$ and $2 \le s \le r - 1$. Hence $i_s \le i_{r-1} \le i_r - 1$, and so

$$k_1 + \sum_{j=2}^{r} k_j i_j = n = i_s m \le (i_r - 1)m.$$

In view of the condition (3.3), we deduce that

$$(i_r-1)m-\sum_{j=2}^{r}k_ji_j \geq k_1 \geq [(m-1)(i_r-1)-1]\sum_{j=2}^{r}k_j+1$$

which can be rewritten as

$$1+\sum_{j=2}^{r-1}k_ji_j+(i_r-1)m\left(\sum_{j=2}^{r}k_j-1\right)\leq i_r\sum_{j=2}^{r-1}k_j.$$

Clearly,

$$\sum_{j=2}^{r}k_j-1\geq\sum_{j=2}^{r-1}k_j,$$

so $(i_r-1)m<i_r$ and thus $i_r<m/(m-1)\leq 2$ which implies $i_r=1$, a contradiction. Thus we conclude that $s=r$. This completes the proof. ∎

For example, the AP-partition (3.1) is uniquely determined by its underlying partition:

$$\{7,9,11\},\ \{8\},\ \{10,12\},\ \{1\},\ \{2,4,6\},\ \{3\},\ \{5\}.$$

We are now ready to present the main result of this paper.

Theorem 3.2. *Given a type $1^{k_1}i_2^{k_2}\cdots i_r^{k_r}$ satisfying the condition* (3.2)*, the number of m-AP-partitions of $\mathbb{Z}_n$ does not depend on m, and is equal to the cyclic multinomial coefficient*

$$\frac{n}{k_1+\cdots+k_r}\binom{k_1+\cdots+k_r}{k_1,\ldots,k_r}. \tag{3.4}$$

In fact, Theorem 3.2 reduces to Theorem 1.1 when we specialize the type to $1^{n-(p+1)k}(p+1)^k$. In this case the condition (3.2) becomes $n\geq kmp+1$. The heads of the k AP-blocks of length $p+1$ satisfy the condition (1.4). Conversely, any k-subset of $\mathbb{Z}_n$ satisfying (1.4) determines an m-AP-partition of the given type. The cyclic multinomial coefficient (3.4) agrees with the formula (1.3) of Theorem 1.1. For example, given the type $1^4 2^1 3^2$ and difference 2, the AP-partition (3.1) is determined by the selection of $\{7,10,2\}$ as heads from $\mathbb{Z}_{12}$.

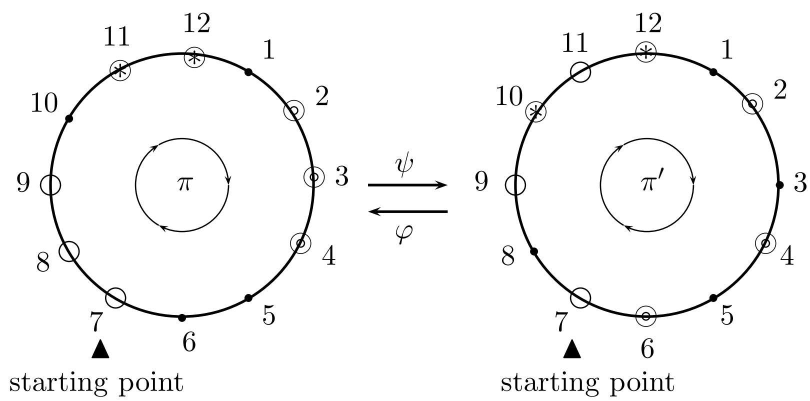
Note that the cyclic multinomial coefficient (3.4) has occurred in Lemma 2.1. Indeed, Lemma 1 is the special case of Theorem 3.2 for $m=1$. We proceed to describe an algorithm, called the *separation algorithm*, to transform m-AP-partitions to m'-AP-partitions of the same type $T=i_1^{k_1}i_2^{k_2}\cdots i_r^{k_r}$, assuming the following condition holds:

$$\left\lceil\frac{k_1}{k_2+\cdots+k_r}\right\rceil\geq(\max\{m,m'\}-1)(i_r-1). \tag{3.5}$$

The separation algorithm enables us to verify Theorem 3.2. We will state our algorithm for m-AP-partitions and m'-AP-partitions, instead of restricting m' to one, because it is more convenient to present the proof by exchanging the role of m and m'.

Given a type $T = 1^{k_1} i_2^{k_2} \cdots i_r^{k_r}$, let $\mathcal{P}_m$ be the set of m-AP-partitions of type T. To prove Theorem 3.2, it suffices to show that there is a bijection between $\mathcal{P}_m$ and $\mathcal{P}'_m$ under the condition (3.5).

Let $\pi \in \mathcal{P}_m$. Denote by $H(\pi)$ the set of heads in π. For each head h of π, we consider the nearest non-singleton head in the counterclockwise direction, denoted h^*. Then we denote by $g(h)$ the number of singletons lying on the path from h^* to h under the convention that h is not counted by $g(h)$. For example, for the AP-partition π' on the right of Figure 2, we have $H(\pi') = \{1, 2, 3, 5, 7, 8, 10\}$, $g(1) = g(3) = g(8) = 0$, $g(2) = g(5) = g(10) = 1$ and $g(7) = 2$. The values $g(h)$ will be needed in the separation algorithm.



$(7,8,9)$, (10), $(11,12)$, (1), $(2,3,4)$, (5), (6) $\quad$ $(7,9,11)$, (8), $(10,12)$, (1), $(2,4,6)$, (3), (5)

Figure 2: The algorithms ψ and φ for $T = 1^4 2^1 3^2$, $m = 1$ and $m' = 2$.

The Separation Algorithm. Let π be an m-AP-partition of type T. As the first step, we choose a head h_1 of π, called the *starting point*, such that $g(h_1)$ is the maximum. Then we impose a linear order on the elements of $\mathbb{Z}_n$ with respect to the choice of h_1:

$$h_1 < h_1 + 1 < h_1 + 2 < \cdots < h_1 - 1 \qquad (\text{mod } n). \tag{3.6}$$

In accordance with the above order, we denote the heads of π by $h_1 < h_2 < \cdots < h_t$, where $t = \sum_{i=1}^{r} k_i$. The m-AP-block of π with head h_i is denoted by B_i. Let l_i be the length of B_i, and so $\sum_{i=1}^{t} l_i = n$.

We now aim to construct m'-AP-blocks $B'_1, B'_2, \ldots, B'_t$ such that B'_i has the same number of elements as B_i. We begin with B'_1 by setting $h'_1 = h_1$ and letting B'_1 be the m'-AP-block of length l_1, namely,

$$B'_1 = (h'_1,\ h'_1 + m',\ \ldots,\ h'_1 + (l_1 - 1)m')\,.$$

Among the remaining elements, namely, those that are not in B_1', we choose the smallest element with respect to (3.6), denoted by h_2', and let B_2' be the m'-AP-block of length l_2 with head h_2'. Repeating the above procedure, as will be justified later, after t steps we obtain an m'-AP-partition, denoted $\psi(\pi)$, of type T with blocks $B_1', B_2', \ldots, B_t'$.

Figure 2 illustrates the separation algorithm from a 1-AP-partition π to a 2-AP-partition π' of the same type $T = 1^4 2^1 3^2$ and vice versa. The solid dots stand for singletons, whereas the other symbols represent different AP-blocks.

We remark that, as indicated by the example, the starting point can never be a singleton. In fact, if s is a singleton and h is a non-singleton head such that all the heads lying on the path from s to h are singletons, then we have the relation $g(h) > g(s)$. Since $g(h_1)$ is maximum, we see that the starting point is always a non-singleton head.

Clearly, it is necessary to demonstrate that the above algorithm ψ is valid, namely, we need to justify that underlying sets of the blocks $B_1', B_2', \ldots, B_t'$ are disjoint.

Proposition 3.3. *The mapping ψ is well-defined, and for any $\pi \in \mathcal{P}_m$, we have $\psi(\pi) \in \mathcal{P}_{m'}$.*

Proof. Let $\pi \in \mathcal{P}_m$ with AP-blocks $B_1, B_2, \ldots, B_t$. Without loss of generality, we may assume that $h_1, h_2, \ldots, h_t$ are the heads of $B_1, B_2, \ldots, B_t$, where h_1 is the starting point for the mapping ψ and $h_1', h_2', \ldots, h_t'$ are the corresponding heads generated by ψ. Let l_i be the length of B_i. Suppose to the contrary that there exist two heads h_i and h_j $(i < j)$ such that

$$h_i' + am' \equiv h_j' + bm' \pmod{n},$$

where $0 \le a \le l_i - 1$ and $0 \le b \le l_j - 1$.

If $a \ge b$, then $0 \le a - b \le l_i - 1$ and $h_j' \equiv h_i' + (a-b)m' \pmod{n}$. But the point $h_i' + (a-b)m'$ is in B_i', contradicting the choice of h_j'. This yields $a < b$ and thus $0 \le b - a \le l_j - 1$.

We claim that the starting point h_1 lies on the path from h_j' to h_i'. In fact, when the Algorithm ψ is at the j-th step to deal with the head h_j, all the points smaller than h_i' lie in one of the blocks $B_1', B_2', \ldots, B_i'$. Then we see that $h_j' > h_i'$. Meanwhile, there are $n - l_1 - l_2 - \cdots - l_{j-1} > 0$ points which are not contained in $B_1', B_2', \ldots, B_{j-1}'$. But the head h_j' is chosen to be the smallest point not in $B_1', B_2', \ldots, B_{j-1}'$, we find that h_j' lies on the path from h_i' to h_1.

In addition to h_i' and h_j', we assume that there are N points on the path from h_j' to h_i'. Since $h_i' \equiv h_j' + (b-a)m' \pmod{n}$ and $1 \le b - a \le l_j - 1$, we obtain $N = (b-a)m' - 1$. On the other hand, at the j-th step, in addition to the point h_j', there are at least $l_j - 1$ points not contained in $B_1', B_2', \ldots, B_{j-1}'$. Similarly, the choice of h_1 and the condition (3.5) yield that the largest $(\max\{m, m'\} - 1)(i_r - 1)$ heads with respect to the order (3.6) are all singletons by the pigeonhole principle. Therefore, there are at least $(\max\{m, m'\} - 1)(i_r - 1)$ points not contained in $B_1', B_2', \ldots, B_{j-1}'$.

It follows that

$$N \geq (\max\{m, m'\} - 1)(i_r - 1) + (l_j - 1). \tag{3.7}$$

Since $N = (b-a)m' - 1$ and $1 \leq b - a \leq l_j - 1$, we deduce that

$$(m' - 1)(i_r - 1) + (l_j - 1) \leq (b-a)m' - 1 \leq (l_j - 1)m' - 1,$$

leading to the contradiction $l_j > i_r$. This completes the proof. ∎

Proposition 3.4. *Given an m-AP-partition of $\mathbb{Z}_n$, the separation algorithm ψ generates the same m'-AP-partition regardless of the choice of the starting point subject to the maximum property.*

Proof. Let π be an m-AP-partition of $\mathbb{Z}_n$. Suppose that $u_1, u_2, \ldots, u_s$ ($s \geq 2$) are all the heads such that $g(u_1) = g(u_2) = \cdots = g(u_s)$ is the maximum on π. Let u_1 be the starting point and $u_1 < u_2 < \cdots < u_s$ with respect to (3.6).

It suffices to show that when the Algorithm ψ processes u_i ($1 \leq i \leq s$), the m'-AP-blocks which have been generated consist of all the elements smaller than u_i. By induction we assume that this statement holds up to u_{j-1}.

Let $v_q, v_{q-1}, \ldots, v_1, u_j$ be all heads lying on the path Q from u_{j-1} to u_j such that $u_{j-1} = v_q < v_{q-1} < \cdots < v_1 < u_j$. Let B_i be the m-AP-block containing v_i. Let l_i be the length of B_i and

$$B'_i = (v'_i, v'_i + m', \ldots, v'_i + (l_i - 1)m')$$

be the corresponding m'-AP-blocks generated by the Algorithm ψ. It suffices to show that the path Q consists of the elements of $B'_s, B'_{s-1}, \ldots, B'_1$.

Suppose that $v_1, v_2, \ldots, v_p$ are all singletons, but v_{p+1} is not a singleton. Then $p \leq q - 1$ since u_{j-1} is always a non-singleton head. The condition (3.5) yields that

$$p \geq (\max\{m, m'\} - 1)(i_r - 1).$$

We now wish to show that for any $1 \leq i \leq q$, the block B_i lies entirely on the path Q. If $i \leq p$, then $B_i = (v_i)$ is a singleton block lying on Q. Otherwise, we have $i \geq p + 1$ and

$$B_i = (v_i, v_i + m, \ldots, v_i + (l_i - 1)m).$$

But the total number of points between any two consecutive elements of B_i is

$$(l_i - 1)(m - 1) \leq (\max\{m, m'\} - 1)(i_r - 1) \leq p.$$

Intuitively, all these points can be fulfilled by the singletons $v_p, v_{p-1}, \ldots, v_1$. Since $u_j > v_1$, the largest element $v_i + (l_i - 1)m$ in the block B_i is smaller than u_j. Hence the block B_i ($i = 1, 2, \ldots, q$) lies entirely on Q.

Therefore, the total number of elements in $B_q, B_{q-1}, \ldots, B_1$ equals the length $u_j - u_{j-1}$ of the path Q. Since B'_i has the same number of elements as B_i, the total number of elements in $B'_q, B'_{q-1}, \ldots, B'_1$ also equals $u_j - u_{j-1}$.

Moreover, it can be shown that the block B'_i also lies entirely on the path Q for any $1 \leq i \leq q$. If $i \leq p$, the block $B'_i = (v'_i)$ is a singleton given by the separation algorithm. Since the total number of elements in $B'_q, B'_{q-1}, \ldots, B'_{i+1}$ is smaller than $u_j - u_{j-1}$ and v'_i is chosen to be the smallest element which is not in $B'_q, B'_{q-1}, \ldots, B'_{i+1}$, we see the relation $v'_i < u_j$. Otherwise, we have $i \geq p+1$ and the total number of points between any two consecutive elements of B'_i equals

$$(l_i - 1)(m' - 1) \leq (\max\{m, m'\} - 1)(i_r - 1) \leq p.$$

Intuitively, all these points can be fulfilled by the singletons $v'_p, v'_{p-1}, \ldots, v'_1$. Since $u_j > v'_1$, the largest element $v'_i + (l_i - 1)m'$ in the block B'_i is smaller than u_j. Consequently, the block B'_i lies entirely on Q.

In summary, the total number of elements in $B'_q, B'_{q-1}, \ldots, B'_1$ which lie on the path Q coincides with the length of Q. Hence the path Q consists of the elements of $B'_s, B'_{s-1}, \ldots, B'_1$. This completes the proof. ∎

Theorem 3.5. *Let T be a type as given before. The separation algorithm induces a bijection between $\mathcal{P}_m$ and $\mathcal{P}_{m'}$ under the condition* (3.5)*.*

Proof. We may employ the separation algorithm by interchanging the roles of m and m' to construct an m-AP-partition from an m'-AP-partition, and we denote this map by φ. We aim to show that φ is indeed the inverse map of ψ, namely, $\varphi(\psi(\pi)) = \pi$ for any $\pi \in \mathcal{P}_m$.

Let $h_1, h_2, \ldots, h_t$ be the heads of π for the map ψ, where h_1 is the starting point. Assume that π has AP-blocks $B_1, B_2, \ldots, B_t$ with h_i being the head of B_i. Let l_i be the length of B_i. By the construction of ψ, the generated heads $h'_1 = h_1, h'_2, \ldots, h'_t$ have the order $h'_1 < h'_2 < \cdots < h'_t$ in accordance with $h_1 < h_2 < \cdots < h_t$. It follows that $g(h'_1)$ is the maximum considering all heads of the AP-partition $\psi(\pi)$.

We now apply the map φ on the m'-AP-partition $\psi(\pi)$ and choose h'_1 as the starting point. Let $h''_1, h''_2, \ldots, h''_t$ be the heads generated by φ respectively. In light of the construction of φ, we have $h''_1 = h'_1 = h_1$ and $h''_1 < h''_2 < \cdots < h''_t$.

For any i, the separation algorithm has the property that the length of the m-AP-block in $\varphi(\psi(\pi))$ containing h''_i is l_i, which is the length of the m-AP-block in π containing h_i.

Note that both $\varphi(\psi(\pi))$ and π are m-AP-partitions. They have the same starting point $h''_1 = h_1$ and the same length sequence $(l_1, l_2, \ldots, l_t)$. Thus for any $i = 2, 3, \ldots, t$, the head h''_i is the smallest point which is not contained in the m-AP-blocks $B_1, B_2, \ldots, B_{i-1}$, and so does h_i. Hence we conclude that $h''_i = h_i$ and $\varphi(\psi(\pi)) = \pi$. This completes the proof. ∎

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